On the theory of Finsler connections especially their equivalence

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§ 1. Introduction

It is well known that the solution of the equivalence problem of linear connections can be formulated with help of the torsion and curvature tensors of the connection (cf. [1] pp. 74—78, [4] p. 77.).

In the case of Varga's theory of line-element connections the conditions of equivalence is formulated with help of a set of tensors differing from the torsion and curvature tensors defined by the structure equations [3]. The Matsumoto's Finsler connection theory [2] can be regarded as a generalization of Varga's theory and hence it seems to be important to discuss the equivalence theory in this case too.

The purpose of our paper is to deal with the equivalence problem using Varga's method.

The terminology of Matsumoto's monograph [2] will be used throughout.

§ 2. Preliminaries

Let $T(M)$ be the tangent bundle of a differentiable manifold M. A non-linear connection N defined by a distribution $y \in TM \to N_y$ in $T(TM)$ satisfying $T_y TM = N_y \oplus T_y^v$, namely the tangent space $T_y TM$ and the vertical subspace T_y^v. Let $L(M)$ be the linear frame bundle of M. The Finsler bundle is defined by $\pi_T^{-1} L(M)$, where π_T is the projection map of $T(TM)$,

A non-linear connection N determines the connection map $K\colon TTM \to TM$ so that for any $z \in T_y TM$ the vertical lift $l^v K(z)$ of the vector $K(z) \in T_{\pi_T(y)} M$ is the vertical component of z. The kernel of this map K is the horizontal subspace N_y of $T_y TM$.

A Finsler connection is a pair (Γ, N) of a connection Γ in the Finsler bundle $F(M)$ (called the *directional connection*) and a non-linear connection N in the tangent bundle $T(M)$.

There are three characteristic linear forms and a vector field on the Finsler bundle, namely the *horizontal basic form* or *h-basic form*

$$\theta_u^{(h)} = u^{-1} \circ d\pi_T \circ d\pi_F,$$

which is independent from the Finsler connection (π_F is the projection in the Finsler bundle $F(M)$),

the *vertical basic form* or *v-basic form*

$$\theta_u^{(v)} = u^{-1} \circ K \circ d\pi_F,$$

which is determined by the non-linear connection N,

the *connection form* ω, which is determined by the directional connection Γ in $F(M)$, and

the *supporting element* $\varepsilon_u = u^{-1} \circ \pi_F$. (Here $u \in F(M)$ is considered as a linear mapping of R^n onto $T_x M$, where $x = \pi_T \circ \pi_F(u)$.)

The values of $\theta^{(h)}$, $\theta^{(v)}$ and ε are in the vector space R^n and the values of the connection form ω are in the Lie algebra $\mathfrak{gl}(n)$.

Let be U a coordinate neighborhood in M with a local coordinate system $x^1, \dots, x^n$. Let $y^1, \dots, y^n$ and $z_1^1, \dots, z_n^1, z_1^2, \dots, z_{n-1}^n, z_n^n$ be the coordinates of the tangent vectors and the liner frames, with respect to the frame $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$. So we have $(x^1, \dots, x^n; y^1, \dots, y^n; z_1^1, \dots, z_n^n)$ as local coordinates in $\pi_F^{-1} \circ \pi_T^{-1}(U) \subset F(M)$. In terms of this coordinate system the forms $\theta^{(h)}$, $\theta^{(v)}$, ω and the vector field ε can be expressed as follows:

$$\theta^{(h)} = \theta^{(h)a} e_a = (z^{-1})_i^a \, dx^i e_a, \tag{1}$$

$$\theta^{(v)} = \theta^{(v)a} e_a = (z^{-1})_i^a (dy^i + N_j^i \, dx^j) e_a = (z^{-1})_i^a \delta y^i e_a, \tag{2}$$

$$\omega = \omega_b^a E_a^b = (z^{-1})_i^a (dz_b^i + z_b^j F_j{}^i{}_k \, dx^k + z_b^j C_j{}^i{}_k \delta y^k) E_a^b, \tag{3}$$

$$\varepsilon = \varepsilon^a e_a = (z^{-1})_i^a y^i e_a. \tag{4}$$

Here $e_1, \dots, e_n$ is a basis for R^n, $E_1^1, \dots, E_n^n$ is a basis for $\mathfrak{gl}(n)$, $(z^{-1})_i^a$ are the elements of the matrix inverse to z_b^i and we used the notation

$$\delta y^i = dy^i + N_j^i \, dx^j.$$

§ 3. Characteristic tensors

The invariants (torsions and curvatures) of the Finsler connection are defined by the structure equations (cf. in an equivalent form [2], 18 §.)

(5) $$d\theta^{(h)a} = -\omega^a_b \wedge \theta^{(h)b} + (z^{-1})^a_l (\tfrac{1}{2} T_i{}^l{}_k \, dx^k \wedge dx^i + C_i{}^l{}_k \delta y^k \wedge dx^i),$$

(6) $$d\theta^{(v)a} = -\omega^b_a \wedge \theta^{(v)b} + (z^{-1})^a_l (P_i{}^l{}_k \, dx^k \wedge \delta y^i + \tfrac{1}{2} S_i{}^l{}_k \delta y^k \wedge \delta y^i + \tfrac{1}{2} R_i{}^l{}_k \, dx^k \wedge dx^i),$$

(7) $$d\omega^a_b = -\omega^a_c \wedge \omega^c_b + (z^{-})^a_i z^j_b (\tfrac{1}{2} R_j{}^i{}_{mk} \, dx^k \wedge dx^m + P_j{}^i{}_{mk} \, dx^k \wedge \delta y^m + \tfrac{1}{2} S_j{}^i{}_{mk} \delta y^k \wedge \delta y^m)$$

as follows:

h-torsion tensors:

(8a) $$T_i{}^l{}_k = F_i{}^l{}_k - F_k{}^l{}_i, \quad C_i{}^l{}_k;$$

v-torsion tensors:

$$R_i{}^l{}_k = \frac{\partial N^l_i}{\partial x^k} - \frac{\partial N^l_i}{\partial y^m} N^m_k - \frac{\partial N^l_k}{\partial x^i} + \frac{\partial N^l_k}{\partial y^m} N^m_i,$$

(8b) $$P_i{}^l{}_k = F_i{}^l{}_k - \frac{\partial N^l_k}{\partial y^i},$$

$$S_i{}^l{}_k = C_i{}^l{}_k - C_k{}^l{}_i;$$

curvature tensors:

$$R_j{}^i{}_{mk} = \frac{\partial F_j{}^i{}_k}{\partial x^m} - \frac{\partial F_j{}^i{}_k}{\partial y^r} N^r_m + F_j{}^r{}_k F_r{}^i{}_m - \frac{\partial F_j{}^i{}_m}{\partial x^k} + \frac{\partial F_j{}^i{}_m}{\partial y^r} N^r_k - F_j{}^r{}_m F_r{}^i{}_k + C_j{}^i{}_r R_k{}^r{}_m,$$

(9) $$P_j{}^i{}_{mk} = \frac{\partial C_j{}^i{}_m}{\partial x^k} - \frac{\partial C_j{}^i{}_m}{\partial y^r} N^r_k - C_r{}^i{}_m F_j{}^r{}_k + C_j{}^r{}_m F_r{}^i{}_k - C_j{}^i{}_r \frac{\partial N^r_k}{\partial y^m} - \frac{\partial F_j{}^i{}_k}{\partial y^m},$$

$$S_j{}^i{}_{mk} = \frac{\partial C_j{}^i{}_k}{\partial y^m} + C_r{}^i{}_m C_j{}^r{}_k - \frac{\partial C_j{}^i{}_m}{\partial y^k} - C_r{}^i{}_k C_j{}^r{}_m.$$

The deflection tensors

(10) $$D^i_k = F_0{}^i{}_k - N^i_k \quad \text{and} \quad C_0{}^i{}_k$$

are defined by the equations

$$d\varepsilon^a = -\omega^a_b \varepsilon^b + z^{-1}{}^a_i (\delta y^i + D^i_k \, dx^k + C_0{}^i{}_k \delta y_k).$$

§ 4. Finsler connections on line-element manifolds

The positive homogeneity of a Finsler connection is a very important property from the standpoint of the geometry of line-element manifolds [3]. In the following we shall characterize the positive homogeneous Finsler connections with help of the torsion and curvature tensors.

Definition 1. (MATSUMOTO) We say that the Finsler connection (Γ, N) is *strictly positively homogeneous* if the local parameters $F_i{}^j{}_k$, $C_i{}^j{}_k$, N_k^j of the connection (Γ, N) satisfy the following conditions in any coordinate system:

a) $$C_i{}^j{}_k(x, y), \quad F_i{}^j{}_k(x, y), \quad N_k^j(x, y)$$

are homogeneous functions with respect to the variables $y^1, \ldots, y^n$ of degree -1, 0, 1, respectively,

b) $$C_i{}^j{}_0 = 0.$$

Definition 2. (cf.[3]). We say that the Finsler connection (Γ, N) is of *Varga type* if it is strictly positive homogeneous and the local parameters $F_i{}^j{}_k$, $C_i{}^j{}_k$, N_k^j of the connection (Γ, N) satisfy the following condition in any coordinate system:

$$N_k^j = F_0{}^j{}_k.$$

Theorem 1. *Let be N a positively homogeneous non-lienar connection. Then the Finsler connection (Γ, N) is strictly positively homogeneous if and only if the following identities are fulfilled*

$$0 = C_j{}^i{}_0 = S_j{}^i{}_{0l} = P_j{}^i{}_{0l}.$$

Proof. Since $C_j{}^i{}_0 = 0$ we can write

$$S_j{}^i{}_{0k} = \frac{\partial C_j{}^i{}_k}{\partial y^m} y^m - \frac{\partial C_j{}^i{}_m}{\partial y^k} y^m = \frac{\partial C_j{}^i{}_k}{\partial y^m} y^m - \frac{\partial C_j{}^i{}_0}{\partial y^k} + C_j{}^i{}_k = \frac{\partial C_j{}^i{}_k}{\partial y^m} y^m + C_j{}^i{}_k.$$

Hence from the identity $S_j{}^i{}_{0k} = 0$ it follows the homogeneity of degree -1 of the functions $C_j{}^i{}_k$.

The curvature tensor $P_i{}^j{}_{kl}$ can be written in the form

$$P_i{}^j{}_{kl} = C_i{}^j{}_{k|l} + C_i{}^j{}_r \left(F_k{}^r{}_l - \frac{\partial N_l^r}{\partial y^k} \right) - \frac{\partial F_i{}^j{}_l}{\partial y^k},$$

where $C_i{}^j{}_{k|l}$ denotes the h-covariant derivative of the tensor $C_i{}^j{}_k$ *). It follows that

$$P_i{}^j{}_{0l} = C_i{}^j{}_{k|l} y^k + C_i{}^j{}_r \left(F_0{}^r{}_l - \frac{\partial N_l^r}{\partial y^k} y^k \right) + \frac{\partial F_i{}^j{}_l}{\partial y^m} y^m =$$

$$= C_i{}^j{}_{0|l} - C_i{}^j{}_k (y^k{}_{|l}) + C_i{}^j{}_r \left(F_0{}^r{}_l - \frac{\partial N_l^r}{\partial y^k} y^k \right) + \frac{\partial F_i{}^j{}_l}{\partial y^m} y^m.$$

*) $T_j^i{}_{|k} = \dfrac{\partial T_j^i}{\partial x^k} - \dfrac{\partial T_j^i}{\partial y^m} N_k^m + T_j^m F_m{}^i{}_k - T_m^i F_j{}^m{}_k$ for instance.

Since $C_i{}^j{}_0=0$, the functions N_j^i are homogeneous and $y^k{}_{|l}=F_0{}^k{}_l-N_l^k$, we get

$$P_i{}^j{}_{0l} = \frac{\partial F_i{}^j{}_l}{\partial y^m} y^m,$$

and thus from $P_i{}^j{}_{0l}=0$ it follows the homogeneity of degree 0 of the functions $F_i{}^j{}_l$.

The inverse part of the theorem is obvious.

Theorem 2. *The Finsler connection (Γ, N) is of Varga type if and only if the following identities are satisfied:*

$$0 = D_k^i = C_j{}^i{}_0 = S_j{}^i{}_{0l} = P_j{}^i{}_0 = 0.$$

Proof. The identity $S_j{}^i{}_{0l}=0$ is equivalent to the homogeneity of the functions $C_j{}^i{}_k$ as above.

Since $D_k^i=0$ we have

$$0 = P_j{}^i{}_0 = F_0{}^i{}_j - \frac{\partial N_j^i}{\partial y^k} y^k = N_j^i - \frac{\partial N_j^i}{\partial y^k} y^k,$$

that is the functions N_j^i are homogeneous of degree 1. As a consequence of $D_k^i=0$ we get the homogeneity of the functions $F_i{}^j{}_k$ too.

The inverse statement is obvious.

§ 5. On the equivalence problem

Let be given two Finsler connections by the connection parameters $\{N_k^i, F_j{}^i{}_k, C_j{}^i{}_k\}$ and $\{\tilde{N}_c^a, \tilde{F}_b{}^a{}_c, \tilde{C}_b{}^a{}_c\}$ in a coordinate neighborhood U in the manifold M with coordinates $x^1, \dots, x^n$. We say that these Finsler connections are locally equivalent if there exists an invertible point transformation $x^i=x^i(\bar{x}^1, \dots, \bar{x}^n)$ which carries the one connection into the other, i.e.

$$\tilde{N}_c^a = N_k^i \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial x^k}{\partial \bar{x}^c} + \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial^2 x^i}{\partial \bar{x}^c \partial \bar{x}^d} \bar{y}^d,$$

$$\tilde{F}_b{}^a{}_c = F_j{}^i{}_k \frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial x^k}{\partial \bar{x}^c} + \frac{\partial^2 x^i}{\partial \bar{x}^b \partial \bar{x}^c} \frac{\partial \bar{x}^a}{\partial x^i},$$

$$\tilde{C}_b{}^a{}_c = C_j{}^i{}_k \frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial x^k}{\partial \bar{x}^c},$$

where $x^1, \dots, x^n; y^1, \dots, y^n$ are the induced coordinates in TU, and $\bar{x}^a=\bar{x}^a(x^1, \dots, x^n)$ denotes the inverse point transformation of $x^i=x^i(\bar{x}^1, \dots, \bar{x}^n)$.

We can formulate the conditions of equivalence as follows.

Theorem 3. *The Finsler connections* $\{N_k^i, F_j{}^i{}_k, C_j{}^i{}_k\}$ *and* $\{\tilde{N}_c^a, \tilde{F}_b{}^a{}_c, \tilde{C}_b{}^a{}_c\}$ *given in a coordinate system* $x^1, \dots, x^n; y^1, \dots, y^n$ *are equivalent if and only if the following mixed system of differential equations is integrabile*

$$(11)\quad (a)\ \frac{\partial x^i}{\partial \bar{x}^a} = p_a^i,$$

$$(b)\ \frac{\partial y^i}{\partial \bar{x}^a} = p_c^i \tilde{N}_a^c - N_k^i p_a^k,$$

$$(c)\ \frac{\partial p_a^i}{\partial \bar{x}^b} = p_s^i \tilde{F}_a{}^s{}_b - F_k{}^i{}_l p_a^k p_b^l,$$

$$(d)\ \frac{\partial x^i}{\partial \bar{y}^a} = 0,$$

$$(e)\ \frac{\partial y^i}{\partial \bar{y}^a} = p_a^i,$$

$$(f)\ \frac{\partial p_a^i}{\partial \bar{y}^b} = 0,$$

$$(12)\qquad \tilde{C}_b{}^a{}_c p_a^i = \tilde{C}_k{}^i{}_l p_b^k p_c^l$$

for the unknown functions $x^i(\bar{x}^1, \dots, \bar{x}^n; \bar{y}^1, \dots, \bar{y}^n)$, $y^i(\bar{x}^1, \dots, \bar{x}^n; \bar{y}^1, \dots, \bar{y}^n)$, $y^i = y^i(\bar{x}^1, \dots, \bar{x}^n; \bar{y}^1, \dots, \bar{y}^n)$, $p_a^i = p_a^i(\bar{x}^1, \dots, \bar{x}^n; \bar{y}^1, \dots, \bar{y}^n)$, *where* $i, a = 1, \dots, n$, *and for the solution we have*

$$\det(p_a^i) \neq 0.$$

§ 6. The conditions of integrability

In the present section we shall determine the conditions of integrability of the mixed system (11), (12) ((11) $(a-f)$ are differential equiatons, (12) is a scalar relation) (cf. [1] pp. 14—18, [4] p. 73.).

It is easy to see that the commutation relations

$$\frac{\partial^2 x^i}{\partial \bar{x}^a \partial \bar{y}^b} = \frac{\partial^2 x^i}{\partial \bar{y}^b \partial \bar{x}^a}, \quad \frac{\partial^2 x^i}{\partial \bar{y}^a, \partial \bar{y}^b} = \frac{\partial^2 x^i}{\partial \bar{y}^b \partial \bar{y}^a}, \quad \frac{\partial^2 y^i}{\partial \bar{y}^a \partial \bar{y}^b} = \frac{\partial^2 y^i}{\partial \bar{y}^b \partial \bar{y}^a}$$

and $\dfrac{\partial^2 p_a^i}{\partial \bar{y}^c \partial \bar{y}^d} = \dfrac{\partial^2 p_a^i}{\partial \bar{y}^d \partial \bar{y}^c}$ are trivially satisfied.

Applying the equations (a) and (c) we can deduce from the commutation relation $\dfrac{\partial^2 x^i}{\partial \bar{x}^a \partial \bar{x}^b} = \dfrac{\partial^2 x^i}{\partial \bar{x}^b \partial \bar{x}^a}$ the condition

$$p^i_c \tilde{T}_a{}^c{}_b = T_k{}^i{}_l p^k_a p^l_b. \tag{13}$$

From the relation $\dfrac{\partial^2 y^i}{\partial \bar{x}^a \partial \bar{y}^b} = \dfrac{\partial^2 y^i}{\partial \bar{y}^b \partial \bar{x}^a}$ we can be deduce

$$p^i_c \tilde{P}_a{}^c{}_b = P_k{}^i{}_l p^k_a p^l_b \tag{14}$$

by using the equations (b), (c), (e), (f).

From the relation $\dfrac{\partial^2 p^i_a}{\partial \bar{x}^c \partial \bar{y}^b} = \dfrac{\partial^2 p^i_a}{\partial \bar{y}^b \partial \bar{x}^c}$ we derive the equation

$$p^i_s \frac{\partial \tilde{F}_a{}^s{}_b}{\partial y^c} = \frac{\partial F_j{}^i{}_k}{\partial y^l} p^j_a p^k_b p^l_c \tag{15}$$

using (e) and (f).

From the commutation relation $\dfrac{\partial^2 y^i}{\partial \bar{x}^a \partial \bar{x}^b} = \dfrac{\partial^2 y^i}{\partial \bar{x}^b \partial \bar{x}^a}$ we get by the use of the equations (a), (b) and (c)

$$p^i_s \left(\tilde{F}_c{}^s{}_a \tilde{N}^c_b + \frac{\partial \tilde{N}^s_b}{\partial \bar{x}^a} - \tilde{F}_c{}^s{}_b \tilde{N}^c_a - \frac{\partial \tilde{N}^s_a}{\partial \bar{x}^b} \right) - P_r{}^i{}_t p^r_c p^t_a \tilde{N}^c_b + P_r{}^i{}_t p^r_c p^t_b \tilde{N}^c_a +$$

$$+ N^i_k (p^k_s \tilde{T}_a{}^s{}_b - T_l{}^k{}_m p^l_a p^m_b) = R_k{}^i{}_m p^m_a p^k_b.$$

Now we apply the relations (13) and (14), so we have

$$p^i_c \tilde{R}_a{}^c{}_b = R_k{}^i{}_l p^k_a p^l_b. \tag{16}$$

At the end the commutation relation $\dfrac{\partial^2 p^i_a}{\partial \bar{x}^c \partial \bar{x}^b} = \dfrac{\partial^2 p^i_a}{\partial \bar{x}^b \partial \bar{x}^c}$ yields the equation

$$p^i_s \tilde{T}_a{}^s{}_{bc} = T_k{}^i{}_{lm} p^k_a p^l_b p^m_c, \tag{17}$$

using (b), (c) and the relations (13), (15), where

$$T_k{}^i{}_{lm} \overset{\text{def}}{=} \frac{\partial F_k{}^i{}_l}{\partial x^m} - \frac{\partial F_k{}^i{}_l}{\partial y^r} N^r_m + F_r{}^i{}_m F_k{}^r{}_l - \frac{\partial F_k{}^i{}_m}{\partial x^l} + \frac{\partial F_k{}^i{}_m}{\partial y^r} N^r_l - F_r{}^i{}_l F_k{}^r{}_m$$

is the Varga's main curvature tensor (cf. [4], p. 13.).

We proved the following

Theorem 4. *A necessary and sufficient condition for the equivalence of two Finsler connections*

$$\{F_j{}^i{}_k, C_j{}^i{}_k, N^i_k\} \quad \textit{and} \quad \{\tilde{F}_b{}^a{}_c, \tilde{C}_b{}^a{}_c, \tilde{N}^a_c\} \quad \textit{is}$$

the existence of a whole number N such that the first N sets of equations in the sequence of sets of equations, which embody the laws of transformations (13)—(17) *of the tensors*

$$C_i{}^j{}_k,\quad T_i{}^j{}_k,\quad P_i{}^j{}_k,\quad \frac{\partial F_i{}^j{}_k}{\partial y^l},\quad R_i{}^j{}_k \quad \text{and} \quad T_i{}^j{}_{kl} \quad \text{and}$$

of the succesive h-covariant derivatives and ordinary derivatives by y^r, shall be compatibile equations for the variables x^i, y^i, p^i_j as functions of the independent variables $\bar{x}^a$, $\bar{y}^a$, and all solutions of these equations shall satisfy the $(N+1)$st set of equations in the sequence.

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